

Superluminal Caustic is Just a Common Misconception: A Comment on astro-ph/0001199 by Zheng Zheng and Andrew Gould

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ABSTRACT

When angular objects in lensing are considered as linear objects, interesting phenomena start happening. Tachyonic caustics are one example. We review that the intrinsic variables of the lens equation are angular variables. We argue that the “fast glance effect” of a caustic curve that is far away from lenses does not share the physical bearing of the well-known (apparent) superluminal motion. There is no doubt that it would be a useful exercise to study the null geodesics in the metric of, say, a rapidly rotating black hole binary. Lienard-Wiechert potentials (A_μ) satisfy Maxwell’s equations in Minkowski space. Authors’ claim that swapping eQ and GM makes the time component (A_0) of the Lienard-Wiechert potentials into “the gravitational analog” that governs the behavior of the null geodesics near a relativistic binary system seems to be unfounded.

Subject headings: gravitational lensing - binary systems

It has been a long controversy in the smoky backroom where non-smoking jurors shred papers and throw verdicts, where the caustics are. The controversy hits home often because we are looking for Bruno’s planets. Caustics matter greatly in the field of microlensing planet searches. We wrote in a paper on the discovery of evidence of a low mass planet (Rhie et al. 2000) that ”a single lens (stellar lens only) has a point caustic at the position of the lens.” Or, is the point caustic behind the position of the lens at the projected position of the lens? May we draw the caustic curve and the critical curve in the same plane as we usually do? Is it a law or a rule that caustics (onto which the critical curve in the image plane is mapped under the lens equation) lie in the source plane? Or, is it a matter of definitions and conventions? What is the image plane? What is the source plane? Does any of them (including what we wrote in the paper mentioned) have any merit or physical relevance? What is a superluminal caustic? We realize that we just found in the phantom of tachyonic caustics an important clue to the mysterious misunderstanding behind the controversy. Discoveries advance science. So does reasoning. Here we investigate the misconception of superluminal caustics (Zheng and Gould 2000) as an attempt to straighten out the wrinkles on caustics. We only need to borrow a pinch of salt from a way of thinking in science which may have been popularized by Einstein: gedenken experiment.

• Much Ado About Nothing

Andrew Gould writes in a recent article (Gould 2000) (G1421 from here on) that “the geometry of point-lens microlensing is so simple that students can derive all the basic results in a few hours.” The abstract of G1421 starts with a paragraph, “if the standard microlensing geometry is inverted so that the Einstein ring is projected onto the observer plane rather the source plane,” What could be the source plane and the observer plane referred to as in G1421? In lensing, there are three basic objects, which we may refer to as “lensing trio” in this article. They are radiation an emission source, a lensing object, and an observer (Schechter 1999). The lensing trio fall more or less on a line in the radial direction, namely, the line of sight (of the observer of the source star in the absence of the intervening lensing object). Given a geometric line, one can imagine an infinitely many planes that are perpendicular to the line. One may refer to the plane that passes through the radiation emission source as the source plane and the plane that passes through the observer as the observer plane. Where is the Einstein ring? Equation (1) in G1421 indicates that the Einstein ring lies on the plane that passes through the lensing object. Zheng and Gould (2000; ZG1199 from here on) refer to the plane through the lensing object as the lens plane.

Physics lies in relations not in nomenclatures, and it will be most harmonious if the nomenclatures faithfully represent the relations. One of the governing relations in lensing is the so-called lens equation, and ZG1199 writes the lens equation in terms of the variables defined on the aforementioned source plane and lens plane (Schneider and Weiss 1986).

$$\eta = \frac{D_s}{D_1} \zeta - D_2 \Theta(\zeta) , \quad (1)$$

where D_1 and D_2 are the distances from the lens to the observer and to the source, and $D_s = D_1 + D_2$. ZG1199 describes the transverse (or 2-d) position variables η and ζ as follows: A photon comes from point

η in the source plane and hits point ζ in the lens plane. At this point, one may wonder if η must be the variable for the source position (since it is said to be in the source plane) and ζ must be the variable for the lens position (since it is said to be in the lens plane). In order to understand the significance of the variables η and ζ , we consider a gedanken experiment: we reduce the mass M gradually by taking away one atom at a time. A photon from point η on the plane at a distance D_s from the observer must hit point ζ on the lens plane such that lens equation (1) with decreasing mass $M(t)$ is satisfied. When the last atom is taken away, there is no lensing mass, and there is no lens plane. The plane at a distance D_1 from the observer is just one of the infinitely many planes that are perpendicular to the line of sight. On the other hand, D_1 retains its significance in equation (1). When there is no lensing mass, Θ vanishes, and the lens equation reads as follows.

$$\eta = \frac{D_s}{D_1} \zeta \quad (2)$$

Since there is no lensing mass, the distance D_1 does not have any physical relevance and should not show up in equation (2) with any significance. But it does. We only know that that is where we used to have a lensing mass. Is it some sort of hysteresis? Then, it could be where we are thinking of putting a lensing mass because we can start piling atoms the very next moment. Then, of course, it could be just a plane we are thinking of for no reason, perhaps out of boredom. Or, perhaps, ζ is not a most representative variable for the governing equation of the lensing behavior. What equation (2) says is that the transverse vector η on the source plane at a distance D_s from the observer and the transverse vector ζ on an arbitrary plane at a distance D_1 from the observer are parallel and scale with the distances from the observer. What should strike us by now is that the perspective of the observer, that is all there is in the lens equation with zero lens mass. So, we divide the transverse vectors by the distances of the planes, and it becomes very clear what the equation must mean. Let $\vec{\alpha} \equiv \zeta/D_1$ and $\vec{\alpha}_s \equiv \eta/D_s$, then

$$\vec{\alpha} = \vec{\alpha}_s . \quad (3)$$

A radiation source that would be seen at an angular position $\vec{\alpha}_s$ by an observer in a flat space is seen by the observer at an angular position $\vec{\alpha}$ that has the same value as $\vec{\alpha}_s$ when there is nothing to change the light ray from that in the Minkowski space. Then, it is clear that the pair of variables $\{\eta, \zeta\}$ simply are not the proper pair of variables for the lens equation. It is not that one can not write the equation in terms of the variables $\{\eta, \zeta\}$, but that they obscure the conceptual underpinning of the lens equation. It is also clear that the variable $\vec{\alpha}$ or $\zeta = D_1 \vec{\alpha}$ is a variable for the positions of the images not the lenses. This fact seems to have generated another misconception that images should lie on the lens plane. This misconception seems to have produced a corollary that images can be in two different lens planes (when the lensing involves double scattering) as the audience was told by an invited speaker in a recent lensing meeting.

Now, we put back in the lensing mass, say, at D_1 . The (2-d) scattering angle Θ is a function of a dimensionful constant GM as well as of ζ : $\Theta(\zeta; GM)$, and it works out to be dimensionless as it should. So, we can divide equation (1) by D_s to write it in terms of the pair of angular variables $\vec{\alpha}$ and $\vec{\alpha}_s$.

$$\vec{\alpha} = \vec{\alpha}_s + \frac{D}{D_1} \Theta , \quad (4)$$

where D is the reduced distance

$$\frac{1}{D} = \frac{1}{D_1} + \frac{1}{D_2} . \quad (5)$$

A radiation source that would be seen at an angular position $\vec{\alpha}_s$ by an observer in Minkowski space is seen by the observer at an angular position $\vec{\alpha}$ which is shifted from $\vec{\alpha}_s$ by a fraction of the (2-d) scattering angle Θ . The reduced distance is no bigger than the smaller of D_1 and D_2 as is familiar from the reduced mass in mechanics. Thus, $D/D_1 \leq 1$, and the angular shift between $|\vec{\alpha} - \vec{\alpha}_s| \leq |\Theta|$. (The equality holds when $D_1 = 0$, which is not exactly a physical situation. That is because of the hidden assumption of the lens equation that the observer is supposed to be asymptotically far away from the lensing mass as it should be clear in the following section.)

An emblematic analogue to this interpretational issue of the lens equation hinging on “trivial math” and “deeper physics” would be the case of the cosmological constant which may have been the biggest blunder of the man of the 20th century (TIME) but likely a necessity for the millenials. As a desperate effort to stop the universe from expanding without introducing negative density or pressure, Einstein modified his equations in 1917 and introduced a new fundamental constant, the so-called cosmological constant. The new equation read as follows (Einstein 1923; Weinberg 1972).

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu} , \quad (6)$$

where $G_{\mu\nu}$ and $T_{\mu\nu}$ are Einstein and energy-momentum tensors, and $g_{\mu\nu}$ is the metric tensor. The LHS is the geometry, the RHS is the matter content, and the Einstein field equation tells us how the geometry of space time interacts with the content of the matter in the space time. What is curious about the cosmological term is that it does not vanish when the space time is flat. It is a blow to the fundamental notion of Einstein gravity one might have just convinced oneself to accept: gravitational interaction is an experience of the curvature of space time. The cosmological term is a non-curvature term of the geometry that participates in governing the gravity as it is written in the LHS of the gravitational field equation. We may consider the Λ -term as a part of the energy momentum tensor and write the gravitational field equation as follows.

$$G_{\mu\nu} = -8\pi G \left(T_{\mu\nu} - \frac{\Lambda}{8\pi G} g_{\mu\nu} \right) \quad (7)$$

The transition from equation (6) to equation (7) is less than trivial mathematically, but it requires a profound change in the frame of physical understanding. The Λ -term as an energy momentum tensor features negative pressure, which Einstein axiomized to avoid and was one of the very reasons why he devised the Λ -term in equation (6) in the first place. It took developement of Goldstone bosons, renormalization and brief marriage of particle physics and condensed matter physics, Grand Unification Theories (GUT), experience of rich vacuum structures (with endless parade of scalar fields as yet to be discovered or refuted) and phase transitions in the early universe, monopole problem, horizon problem, dark matter problem, inflation theories, structure seed problem, fine-tuning problems, topological defects, etc, for the negative pressure to find its natural position in human intellectual domain. Now, no one doubts the physical relevance of the cosmological constant as the the vacuum energy density – or the “zero point energy” of the (future) quantum gravity, even though it is a good question how big it is, or whether it is a constant. In

astrophysical practice, it is simply passed as a stuff with stiff equation of state: $p = -\rho$ that can overcome the self-gravity of ordinary matter and make the universe fly apart. In fact, high- z supernovae searches have found evidence of accelerating universe (Perlmutter et al. 1997; Schmidt et al. 1998).

So, we emphatically conclude that the intrinsic variables in lens equations in (1) and (4) are angular variables. This obvious conclusion is self-evident from the beginning if we derive the lens equation of a point mass M starting from the first principle (simply following a standard textbook on general relativity). An equation that relates two angular variables, that is the first thing we get. The others are all simple derived quantities, and there is no misunderstanding of what is what. If it takes a few hours for students to do so as Gould (G1421) testifies, it should take less than a few days for practitioners to derive the lens equation and less than a few minutes to read one. So, we write out the derivation in the following section as an effort to abolish the ground for the mysterious understanding that seems to perpetuate even more mysterious controversies. In fact, it is cathartic to go through the derivation of the lens equation once. The Einstein field equations are non-linear and can not be solved in general, but there are some exact solutions. And, the lens equation of a point mass is derived from one of those exact solutions, and that the simplest one (despite the problem of no global time killing vector and of nontrivial topology). It is such a great assurance to be backed by an analytic derivation from an exact solution.

Incidentally, one may realize that the source plane and the lens plane defined based on the positions of the source and the lens in radial direction (or based on the distances of the source and the lens from the observer) have no relevance with the variables in the lens equation. The lens equation can only address the relation between transverse position variables. Then, one wonders why these objects play such a persistent role in the volitional papers and caustics.

Let's pause for a moment and ponder angular variables. What is it that we perceive and measure as the angular position of a celestial object? Speckle imaging may offer the best food for thought. The space time dependent refraction index makes a photon beam from a celestial object wiggle along through the atmosphere, and the object appears to hop around as registered at the focal plane. We may refer to these snap shots as a time series of “apparent angular positions” of the object. If we remove the atmospheric turbulence or the atmosphere altogether (again in our thought experiment) – assuming that that leaves only the vacuum for the photon beam to propagate through, we will find a steady image on our CCD. We may refer to it as the “true angular position” of the object. What is common for both “apparent” and “true” angular positions is that an angular position is determined by the direction of the propagation vector of the photon beam at the observer. The space time dependent refraction index of the atmosphere is an electromagnetic property of a matter in hydrodynamic motion. On the other hand, there is nothing wrong with understanding gravitational lensing effect of the space time curvature in terms of effective refraction index (continuous function in space and time) assuming that we calculate the index truthfully to the underlying physics. That is, according to general relativity, not in terms of Fermat theorem (Gould and Loeb 1992) unsubstantiated for the gravitational effect on optical paths (no historians have found evidence of a margin the lack thereof prevented Fermat from elaborating Riemannian geometry and propagation of massless spin one particles).

Let's consider a quasar lens. There are four objects that are believed to be all from the same emission source QSO 2237+0305, and they are called Huchra's lens, Einstein cross, or QSO 2237+0305. The emission source QSO 2237+0305 is at a cosmological distance $z = 1.695$, and it will be a long time before the lensing galaxy moves away from the line of sight of the QSO even though the (Sb) galaxy is relatively close to us at $z = 0.0394$. So, the four objects are the thing that will be recognized as QSO 2237+0305 for generations to come, but we always can remove the lensing galaxy in our thought experiment. Then, an observer will see one object, say, at an angular position $\vec{\alpha}_s$, which one may refer to as the “true angular position” of the quasar. In a consistent manner, one might have liked to refer to the angular positions of the four objects as the “apparent angular positions” of the quasar, but the multiplicity of the objects renders such practice seem unfit. The four objects can represent the different facets of the lensed quasar, and they are usually referred to as the “images” of the quasar. So, the angular positions of the four objects may be referred to as the “angular positions of the images” of the quasar or simply the “image positions”. This practice is in perfect harmony with our everyday experience. A client in a barber chair next to a corner with mirrored walls can see multiple “images” of oneself while being oblivious to the true object, oneself. In fact, the images show the different sides of the client. So, we are all content to call the four objects the images of the quasar. Now, we repeat our favorite thought experiment and reduce the mass of the lensing galaxy to zero. An observer will see one image of the quasar. What should we call this image? Preimage? Unlensed image? Image-sub-zero (Image₀: the image one sees when the lensing mass vanishes)? Instead, this particular image is usually referred to as the “source”. Then, the “true angular position” of the quasar $\vec{\alpha}_s$ may be referred to as the “angular source position” or simply the “source position”, and equation (3) may read in English as follows: the image position of an object is the same as the source position of the object when there is no lensing mass. So, $\vec{\alpha}$ we introduced as the equivalent of ζ/D_1 is the variable for the image positions. When we put back in the lensing galaxy, the image positions may differ from the source position, and their relation is nothing but the lens equation (4). One of the main games in quasar lensing (or any large scale lensing) is to reconstruct $\vec{\alpha}_s$ and Θ from observational information of the images $\vec{\alpha}$.

In quasar lensing, the distance (actually the redshift) to the emission source is one of the better determined quantities. As a consequence, one may favor (Wyithe et al. 2000) to write the lens equation in terms of linear variables by multiplying the equation (4) by the distance to the source D_s (assumed to be determined from the measured redshift and the cosmology to be determined).

$$(D_s \vec{\alpha}_s) = (D_s \vec{\alpha}) - \frac{D}{D_1} (D_s \Theta) \quad (8)$$

In the case of QSO 2237+0305, the variables $(D_s \vec{\alpha}_s)$ and $(D_s \vec{\alpha})$ may be considered to be defined on the plane at $z = 1.695$ from the observer. (The equivalence between angular variables and linear variables holds valid in a small scattering angle approximation, $\Theta \ll 1$, which we assume to be the case in this article. The angular separations of the four objects in QSO 2237+0305 is about $1'' \ll 1$.) One may refer to the plane at $z = 1.695$ as the source plane, as in G1421 (and references therein), and consider $(D_s \vec{\alpha}_s)$ and $(D_s \vec{\alpha})$ as the linear variables projected into the source plane (Wyithe et al. 2000). Then, the variable $(D_s \vec{\alpha}_s)$ denotes the source position in the source plane, and the variable $(D_s \vec{\alpha})$ denotes an image position in the source plane. If one feels a bit of cluttered tautology here, one may realize that the culprit is the clinging desire to recognize the quasar in full three-dimensional coordinates of the space.

The source position has been assigned three coordinates: $(D_s \vec{\alpha}_s, D_s)$, and so has been the position of the image: $(D_s \vec{\alpha}, D_s)$. We note that the images lie on the source plane here. We mentioned before that some practitioners insist on putting images on the lens plane (or even lens planes).

The source plane here is tied to the value of the third (or radial) coordinates D_s in $(D_s \vec{\alpha}_s, D_s)$ and $(D_s \vec{\alpha}, D_s)$. On the other hand, $\{D_s \vec{\alpha}_s\}$ spans a two-dimensional plane, and one may prefer to refer to the plane as the source plane because the plane is parameterized by the source position variable. The two source planes coincide and there doesn't seem to be any conflict between the two definitions. That is, until one realizes that $\{D_s \vec{\alpha}\}$ also spans a two-dimensional plane, and one may prefer to refer to the plane as the image plane because the plane is parameterized by the image position variable. So, we find ourselves in the middle of a luxury of definitions that seem to be tangled in redundancy: the source plane originally defined by the distance of the emission source at D_s and parameterized by the (2-d) source position variable may be preferred to be referred to as the source plane, and the source plane originally defined by the distance of the emission source at D_s and parameterized by the (2-d) image position variable may be preferred to be referred to as the image plane.

Lens equation is a relation of two dimensional variables and can accommodate only two dimensional angular variables or corresponding two dimensional linear variables. Even when one carries around the third (radial) components, the degrees of freedom in the lens equation is only two dimensional. One can choose a plane that may represent the angular space faithfully with an understanding that any plane is as good as any other plane.

$$\begin{aligned} (\vec{\alpha}_s, 1) &= D_s^{-1} (D_s \vec{\alpha}_s, D_s) = D_\xi^{-1} (D_\xi \vec{\alpha}_s, D_\xi) \\ (\vec{\alpha}, 1) &= D_s^{-1} (D_s \vec{\alpha}, D_s) = D_\xi^{-1} (D_\xi \vec{\alpha}, D_\xi) \end{aligned}$$

One may choose the plane at a unit distance, at D_s , or at an arbitrary distance D_ξ from the observer. They are all equivalent. The scattering angle is a quantity defined on an optical path – a one-dimensional object in space. Lensing is defined on the space of optical paths, which one as an observer recognizes only at one end of the paths. Once a plane is chosen, that is where all the lensing variables will be defined and compared. Thus, it is best to consider the chosen plane as the “abstract plane” which may be parameterized by the source position variable or the image position variable. We may prefer to refer to the “abstract plane” as the “abstract lens plane” or simply the “lens plane” because that is where the lens equation is defined and studied. One may refer to the “abstract lens plane” parameterized by the source position variable as the source plane and the “abstract lens plane” parameterized by the image position variable as the image plane. So, the lens equation is a mapping from the “abstract lens plane” to itself, or from the image plane to the source plane.

$$(D_\xi \vec{\alpha}_s) = (D_\xi \vec{\alpha}) - \frac{D}{D_1} (D_\xi \Theta) \quad (9)$$

When $D_\xi = D_1$, the “abstract lens plane” coincides with the lens plane defined by the radial position of the lensing object as in G1421 and ZG1199 (and references therein). As we will see in the following section, the distance from the lensing mass of the apastron of an optical path around a lensing mass is the same as the Einstein ring radius on the plane at D_1 (in the approximation of linear gravity which is valid

for all the observed and identified gravitational lenses). Einstein ring refers to a ring image (as well as the critical curve) in a point mass lens, and there seems to be a (wrongful) religious belief among some practitioners that the plane at D_1 is endowed with a privileged position as the image plane. That is not so, contrary to what one may find in G1421 and ZG1199. It is indeed baffling to hear as recently as July 1999 a claim (Petters 1999) that images can be in two different lens planes. An observer does not see the photons in the images until they arrive at the observer as we have discussed repeatedly. There is only one plane one may define: the “abstract lens plane”, the representation of the observer’s sky – the motherboard of both the image plane and source plane. One can put the “abstract lens plane” anywhere one finds it useful as far as the smallness of the scattering guarantees linearity between the plane and the sky.

The critical curve and the caustic curve pertain to the differential behavior of the lens equation. The lens equation is an explicit mapping from an image position to its source position, and there are multiple solutions for a given source position. The multiplicity can change from domain to domain of the source plane, and all these interesting behavior can be studied starting from differentiating the lens equation.

$$d\vec{\alpha}_s = d\vec{\alpha} - \frac{D}{D_1} \frac{d\Theta}{d\vec{\alpha}} d\vec{\alpha} \quad (10)$$

When one of the (never both in microlensing we are interested in) eigenvalues of this linear transformation, $d\vec{\alpha} \rightarrow d\vec{\alpha}_s$, vanishes, the lens equation is said to be stationary along the eigendirection. The set of the points where the eigenvalue vanishes is called the critical curve. This is a benign or natural generalization of what is familiar from a real function, say, $y = f(x) : x, y \in \mathcal{R}$. Critical points are where $df/dx = 0$ (or $f(x)$ is not differentiable). Sometimes, we may hear “critical line” in relation to lensing. First of all, the set of critical points almost always form closed smooth curves in lensing. (One can assign lensing objects at infinity and force the critical curve to have an open curve, whose physical relevance I am not certain of.) Also, “critical line” may be best left as the terminology for the loci of the zeros of the Riemann Zeta function. (The zeros are believed to be on the “line” whose real value is $1/2$, and this Riemann conjecture remains to be a conjecture despite the telegram sent by Hilbert claiming otherwise.) Thus, we prefer to call the set of critical points of the lens equation the critical curve. The critical curve may be a disjoint sum of closed curves. One may wonder what happens to the critical curve under the mapping of the lens equation. The resulting curve is called the caustic curve. The caustic curve has the same connectedness as the critical curve because the lens equation is continuous (actually smooth) in the neighborhood of the critical curve. Continuity is preserved under a continuous mapping. On the other hand, I have no idea why the caustic curve is named as it is, even though they look punky all right with spiky cusps. In CRC Concise Encyclopedia of Mathematics, caustics are defined as involutes, and involution vaguely reminds me of the way light rays pile up on the glittering surface of a swimming pool on a bright day. Caustic curves I encounter in microlensing are all “some-form-of-oids” similar to cycloids: smooth closed curves punctuated by cusps. Cusps occur because the lens equation has stationary points along the critical curve. A household name example of cusps may be the highest points in the swinging of a pendulum. The trajectory of the pendulum in space changes the direction of the tangents (or the velocity) to the curve at the stationary points where the kinetic energy vanishes. In The Random House College Dictionary, caustic is defined as to be severely critical, sarcastic, or capable of burning living tissue. We find the caustic curves relatively benign or even slightly enjoyable (we can generate relatively interesting looking algebraic curves

from physical necessity!), but the mythology around the caustic curve seems to have been, well, caustic. One wonders whether the smoke in the backroom may be dully tributed as the shroud of acquired memories of Bruno burning at stake wondering of the neurochemistry of the minds of the inquisitors and leaving behind his philosophical conjecture on ubiquitous planets to be scientifically tested some four hundred years later.

So, where are the critical curve and the caustic curve? They are objects defined through the lens equation and all lie in the same space: angular space. Or, an equivalent linear space. We choose a lens plane, that is, an “abstract lens plane”, and mark lens positions, source positions, image positions, Einstein ring, critical curve and caustic curve. And, anything else we may feel useful.

Then, how does a caustic curve fly tachionically or at a speed faster than the speed of light? It doesn't. A caustic curve is not an object physically occupying a space at D_s from the observer. It does not swirl on the plane at D_s from the observer in unison with a pair of binary masses in a relativistic orbital motion at a distance of D_1 from the observer. One may suggest: apparent superluminal motion of blobs in a microquasar is a projection effect, and exactly the same phenomenon happens to the caustic curve once it is projected to a linear space such as the D_s -plane. Does it not? Of course, not. The analogy is flawed. In the case of the blobs from the microquasar, they are the objects out there moving at certain linear velocities, and we can only measure the motions in terms of angular shifts in time. The shifts of the angular positions of the blobs from the angular position of the microquasar which is the emission source of the blobs of particles can correspond to a speed larger than the speed of light when multiplied by the distance to the microquasar. That is referred to as an (apparent) superluminal motion, and it offers an information on the direction of the beam of the blobs. In contrast, the significance of defining superluminal caustic is as substantial as defining superluminal eyes upon having made a sweeping glance at the Milky Way from Ayers Rock. A caustic is nothing more than a peeping hole in this regard. As the caustic curve moves, the target stars that can be sampled through the window defined by the caustic curve change. Furthermore, what we see are images not the caustic curve. The caustic curve as an aperture does not deliver images directly. Images are delivered only after the tranformation dictated by the lens equation is carried out. We will see in a following section that there is no “superluminality” to interest us even when we indulge in the phantom world of tachyonic caustics. Only the effect of fast glance: caustic crossing signal buried under finite size source effect and “long” exposure.

• Lens Equations are Relations of Optical Paths

Schwarzschild metric is an exact solution to the Einstein field equations with a point mass. If the point mass is M , the Schwarzschild metric is given by (Weinberg 1972)

$$ds^2 = - \left(1 - \frac{2GM}{r} \right) dt^2 + \frac{dr^2}{1 - 2GM/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (11)$$

where the Schwarzschild radius $r_s = 2GM$ is 2.95km for a solar mass object. In microlensing, photons' passage is about 10^8 times the Schwarzschild radius. The optical path is found by solving the free fall

equation with the null condition $ds^2 = 0$, and the orbit $\theta(r)$ is given as an elliptic integral that requires numerical estimation. When the closest approach r_o of the photon beam to the mass M is 10^8 times r_s or so, however, the weak gravity allows truncation of the integral at the linear order (in the Newtonian potential GM/r , which is called Robertson expansion), and the scattering angle of the orbit takes a simple form.

$$\Theta = \frac{4GM}{r_o} = \frac{2r_s}{r_o} \quad (12)$$

In Newtonian gravity, the scattering angle is given by twice the value of the Newtonian potential at the closest approach, $2GM/r_o$, and differs from Einstein gravity factor 2. This factor 2 difference was crucial in establishing Einstein theory as the theory of gravity.

In Newtonian gravity, an unbound orbit forms a hyperbolic curve (on the plane defined by an azimuthal angle $\phi = \text{constant}$). If we consider the family of hyperbolic curves connecting two asymptotic points that represent an emission source and an observer, the scattering angle the hyperbolic curve represents grows with the distance from the lensing mass located somewhere between the emission source and the observer. In GR, the photon trajectories (in the Schwarzschild coordinates) are not exactly hyperbolic, but the family of photon trajectories share the same behavior: the scattering angle grows with the distance to the lens position. On the other hand, equation (12) tells us that the scattering angle Θ is inversely proportional to r_o . Therefore, there are only two possible null geodesics from a given emission source to a given observer for a given azimuthal angle.

Figure 1 shows the scattering plane ($\phi = \text{constant}$) and two null geodesics (or optical paths). A photon emitted along the tangent to an optical path at the emission source arrives at the observer with the propagation vector tangent to the optical path at the observer. Thus, the observer sees two stars, one at (α_1, ϕ) and the other at (α_2, ϕ) (in this unorthodox angular position coordinate system). If we remove the lensing mass M (or wait for the lensing mass to move away), the observer will see one star at (α_s, ϕ) . In lensing jargon, (α_s, ϕ) is referred to as the source position, and (α_1, ϕ) and (α_2, ϕ) are referred to as the image positions. The relation between the source position and the image positions is called the lens equation and is obtained easily from the diagram in figure 1. If α is the variable for the image positions, and D_1 and D_2 are the distances from the lens to the observer and to the source along the line of sight, the lens equation is given by

$$\alpha - \alpha_s = \frac{D_2}{D_1 + D_2} \frac{4GM}{D_1 \alpha} \equiv \frac{\alpha_E^2}{\alpha}. \quad (13)$$

This is a quadratic equation and has two solutions for α for each α_s . When $\alpha_s = 0$, the two solutions are reflection symmetric: $\alpha = \pm \alpha_E$. In fact, when $\alpha_s = 0$, the scattering plane is not uniquely determined due to the azimuthal symmetry, and the images form along a ring of radius α_E . This ring is the famous Einstein ring, and α_E is referred to as the angular Einstein ring radius. In the small scattering angle approximation we are using, the closest approach r_o of these photon paths to the lensing mass M is the same as the Einstein ring radius $R_E \equiv D_1 \alpha_E$. D_1 and D_2 ,

$$R_E \equiv \sqrt{4GMD}, \quad (14)$$

where D is the reduced distance of D_1 and D_2 . The lens equation (13) can be written in terms of linear variables. Let $b \equiv D_1 \alpha$ and $s \equiv D_1 \alpha_s$. Then, the variables are defined on the plane that passes through the

lensing mass.

$$b - s = R_E^2 \frac{1}{b} \quad (15)$$

In order to incorporate the variable (ϕ) for the orientation of the scattering angle, we should write it as a (2-d) vector equation.

$$\vec{b} - \vec{s} = R_E^2 \frac{\vec{b}}{b^2} \quad (16)$$

So far, the lensing mass has been at the origin of the coordinate system. If it is at \vec{x} , the lens equation becomes

$$\vec{b} - \vec{s} = R_E^2 \frac{\vec{b} - \vec{x}}{(\vec{b} - \vec{x})^2}. \quad (17)$$

This can be extended to multiple particle lens systems. For a binary lens,

$$\vec{b} - \vec{s} = R_E^2 \left(\frac{\epsilon_1(\vec{b} - \vec{x}_1)}{(\vec{b} - \vec{x}_1)^2} + \frac{\epsilon_2(\vec{b} - \vec{x}_2)}{(\vec{b} - \vec{x}_2)^2} \right), \quad (18)$$

where R_E is the Einstein ring radius of the total mass M , $R_E^2 \equiv 4GM D$, and ϵ_1 and ϵ_2 are the fractional masses located at \vec{x}_1 and \vec{x}_2 respectively.

The two-dimensional vectors are most ideally handled as complex variables. Most of all, that is the only way to solve the binary equation. So, let's complexify the variables: $\vec{b}, \vec{s}, \vec{x}_1, \vec{x}_2 \rightarrow z, \omega, x_1, x_2$. Then equations (16) and (18) are rewritten as follows.

$$\omega = z - \frac{R_E^2}{\bar{z}} \quad (19)$$

$$\omega = z - R_E^2 \left(\frac{\epsilon_1}{\bar{z} - x_1} + \frac{\epsilon_2}{\bar{z} - x_2} \right) \quad (20)$$

We can choose the coordinate system so that the lens position variables x_1 and x_2 are real. Once we introduce a complex variable on the plane on which the lensing variables are defined (so commonly referred to as the lens plane), the lens plane as a two-dimensional linear space is parameterized by the complex variable and its complex conjugate. What is convenient about complex variables is that we only need to write half the equation. For example, equation (19) implies that the following is also true.

$$\bar{\omega} = \bar{z} - \frac{R_E^2}{z} \quad (21)$$

Incidentally, we have defined two sets of variables on the lens plane: One for the image position variable, (z, \bar{z}) , and the other for the source variable position variable, $(\omega, \bar{\omega})$. One may wonder if it is necessary to consider a projected plane to define complex variables. That is not so. We could have defined a complex plane for the angular space parameterized by $\vec{\alpha}(\leftarrow \alpha)$ or $\vec{\alpha}_s(\leftarrow \alpha_s)$. It is just that we have identified the angular space and a projected plane, which is valid because $\Theta \ll 1$. As a matter of fact, we could have chosen any projected plane as our lens plane where we define lensing variables. What is invariable is the fact that the observer sees images and recognizes the tangent of the optical paths at the observer as

the angular positions of the images in the sky. In fact, we do not have to adhere to the linear scale of the projected plane, and it is customary to normalize the equation (or scale the lens plane) so that $R_E = 1$. For example, the binary lens equation can be rewritten in dimensionless variables as follows.

$$\omega = z - \frac{\epsilon_1}{\bar{z} - x_1} - \frac{\epsilon_2}{\bar{z} - x_2} \quad (22)$$

• Off-Axis Trioids of a Binary Lens

Einstein ring is well known, but it is still an interesting object to think about if we think about it. If the radiation emission from the source is a uni-directional coherent beam as in a laser, the observer will be able to see one image of the source at most let alone a ring image. However, many heavenly bodies are largely isotropic radiation emitters. So, when the lensing trio are aligned, the observer sees infinitely many rays from the source, $\{\vec{\alpha} \mid |\vec{\alpha}| = \alpha_E\}$, instead of one, $\{\vec{\alpha} \mid \vec{\alpha} = \vec{\alpha}_s\}$. If we look at the lens equation (19), it is an explicit mapping from an image position to its source position. So, when the lensing trio is aligned, a continuum of image positions is mapped to one source position under the lens equation. In other words, the Einstein ring is the set of stationary points of the lens equation. The curve of stationary points of a mapping seems to be said to be critical (I have an impression that anything in mathematics that may be remotely interesting is said to be critical), hence Einstein ring is a critical curve of a point mass lens. The stationarity is due to the azimuthal (or axial) symmetry of the lensing system, hence the tangent (azimuthal vector) to the Einstein ring vanishes under the lens equation but not the normal. The linear differential behavior of a mapping is conveniently described by the Jacobian matrix $(d\vec{\alpha}_s/d\vec{\alpha})$ written out in a 2 by 2 array, and the criticality shows up as a vanishing eigenvalue of the Jacobian matrix. We differentiate equation (19).

$$\begin{pmatrix} d\omega \\ d\bar{\omega} \end{pmatrix} = \begin{pmatrix} \partial_z \omega & \partial_{\bar{z}} \omega \\ \partial_z \bar{\omega} & \partial_{\bar{z}} \bar{\omega} \end{pmatrix} \begin{pmatrix} dz \\ d\bar{z} \end{pmatrix} \equiv \mathcal{J} \begin{pmatrix} dz \\ d\bar{z} \end{pmatrix} \quad (23)$$

where the Jacobian matrix \mathcal{J} is given as follows.

$$\mathcal{J} = \begin{pmatrix} 1 & \bar{\kappa} \\ \kappa & 1 \end{pmatrix} \quad ; \quad \kappa \equiv \frac{R_E^2}{z^2} \quad (24)$$

The eigenvalues are

$$\lambda_{\pm} = 1 \pm |\kappa| \quad (25)$$

(The eigenvalues are real, which must be expected from that the Jacobian matrix is hermitean : $\mathcal{J}^\dagger = \mathcal{J}$). When $|z| = R_E$, $|\kappa| = 1$, and λ_- vanishes. So does the Jacobian determinant, of course, which is the product of the eigenvalues.

In the case of a binary lens, the differential equations are exactly the same except for that κ is given by

$$\kappa \equiv \frac{\epsilon_1}{(z - x_1)^2} + \frac{\epsilon_2}{(z - x_2)^2} . \quad (26)$$

We have chosen $R_E = 1$ as in the (normalized) binary equation (20). On the critical curve, where $\lambda_- = 0$ and $\lambda_+ = 2$, κ is a pure phase because $|\kappa| = 1$. So, the critical curve is the set of the solutions to the analytic equation (26) with

$$\kappa = e^{2i\varphi} : \quad \varphi \in [0, \pi) . \quad (27)$$

When the separation $\ell \equiv |x_1 - x_2|$ between the two lens elements is smaller than ℓ_- , the critical curve is made of three loops and so is the caustic curve.

$$\ell_- = (\sqrt[3]{\epsilon_1} + \sqrt[3]{\epsilon_2})^{-\frac{3}{4}} ; \quad \frac{1}{\sqrt{2}} \leq \ell_- < 1 \quad (28)$$

So, the caustic curve of a binary lens with $\ell \lesssim 0.7$ is made of three disjointed loops irrelevantly of the fractional mass parameter ϵ_s . Figure 2 shows an example of a (symmetric or equal mass) binary lens with $\ell = 0.55$. The critical curve is in blue, the caustic curve is in red, and the two crosses in black are the positions of the lenses. The line that connects the two lens elements of a binary lens is referred to as the lens axis. The caustic loop with four cusps (quadroid) always crossed the lens axis, and the two triangular caustic loops (trioid) always are off the lens axis. The small critical loop encloses the limit point which is at $z_{*\pm} = \pm i\ell/2$. The corresponding points (correspondence by the mapping of the lens equation) fall inside the trioids.

$$\omega_{*\pm} = \pm i \left(\frac{\ell}{2} - \frac{1}{\ell} \right) \quad (29)$$

Figure 2 shows a source trajectory with one end at ω_{*+} in green and the corresponding image trajectories in magenta. We have chosen this half-way trajectory so that the accidental symmetry due to the equal mass would not mire the visual clarity of the behavior of the image trajectories. The union of the yellow curves and the magenta curves represent the total images of the line source trajectory with $\omega = -1.75i$. Readers are encouraged to be impressed by the similarity between the source trajectory and the image trajectory at the bottom of the plot and the relatively small area the image trajecotries inside the large critical loop occupy. The parity of the images is positive outside the large critical loop and inside the small critical loops. Inside the large critical loop and outside the small critical loops, the images have negative parity. There are usually three images in a binary lens, and the corresponding image trajectories are the one at the bottom of the plot converging to ∞ (positive) and the two outside the small critical loop converging to the lens positions. Since $\omega = -1.75i$ is inside a caustic loop, we expect two more images while the source trajectory remains inside the caustic loop. They are the small segments that connected at a critical point on the small critical loop in the upper half plane.

In the case of a symmetric lens, the two limit points and the lens positions form a square. So, the distance between the two small critical curves is about the same as the separation between the lensing masses, and the caustic crossing images are at a distance of about half the separation from the center of the mass. As ℓ becomes small, the distance of the limit points decrease linearly with ℓ and do all the images but the image near the source. This means that all the images but the image near the source become only nominal images, and that is reflected on the off-axis caustics moving away from the lens axis inversely proportional to the separation ℓ (see equation (29)). Also, the sizes of the critical loops and caustic loops shrink as ℓ shrinks. What it means is that the lensing elements are so close to each other that the binary lens behave more or less as a single lens. If we assume that $\ell = 0.1$ as in ZG1199, the caustic in the lower half-plane will be at $\omega = -9.95i$. The microlensing amplitude of a single lens of a source at a distance of 9.95 (in Einstein ring radius unit) from the lens is 1.000196. So, the effect of lensing on the image near the source is a brightening by $\sim 0.02\%$, which is practically equivalent to no lensing at all. Also, as the trioid caustic shrinks practically to a point at $\omega = -9.95i$, the finite size source effect washes out the singular

brightening effect of the critical curve (the images of the most of the part of the star falls away from the critical curve, and its average falls below any reasonable detection level). The side of the trioids measure about 0.00045 in units of Einstein ring radius. Usually, the Einstein ring radius is $\lesssim 1$ mas, then the size of the trioids will be $\lesssim 0.45\mu\text{as}$. The solar radius at 8 kpc from us will be about $0.565\mu\text{as}$.

Now, let the binary rotate. Let's put the (abstract) lens plane at D_1 . The linear speed of the trioids will be hundred times larger than that of the binary masses. If we assume that the lens is half way to the source that is 8 kpc away from us, then the Einstein ring radius of a solar mass lens is 4 au. If the binary is face on, and $\ell = 0.4$ au, then the orbital velocity is 50 km/sec, and trioids move at an apparent speed of 5,000 km/sec = 0.0167 (in units of the speed of light) which is hardly a relativistic speed. If the lensed star has the solar radius, then it is 1.16 seconds on the (abstract) lens plane. So, it takes about 140 seconds for the trioid to sweep across the solar diameter. If the exposure time is order of a few minutes (with a moderate size telescope), the signal of the caustic crossing will be contained in one frame. If one arranges the apparent speed of the trioid to be bigger, the effect will be to shorten the duration of the signal. There is no physical bearing this “fast glance effect” of a far-away caustic shares with the “superluminal effect” of a particle beam moving at an angle with respect to the line of sight.

• Name That “-OID”

In a gravitational binary lens, one encounters three types of “some-form-of-OID's”: “trioid” with three cusps, “quadroid” with four cusps, and “hexoid” with six cusps borrowing the names from our own paper on line caustic crossing and limb darkening (Rhie and Bennett 1999). The “-OID's” in binary lenses are all simple loops with winding number one unlike in higher multiple point lenses. In an effort to avoid cooking up redundant nomenclatures for “some-form-of-OID's”, we looked up “CRC Concise Encyclopedia of Mathematics” edited by Chapman and Hall (CRC from hereon) with “tricuspid” suggested by an “authority” for “trioid” in mind.

“TricuspidOID” seems to be a mathematical term even though “tricuspid” is not. So is “deltoid”, which seems to originate from the shape of the Greek letter Δ . It is also the anatomic term for a large muscle covering the shoulder joint. We get an impression that anatomy and geometry must have been developed hand in hand. A “nephroid” is an “-oid” with two cusps one can generate using a so-called supercritical lens. It looks somewhat “like a kidney” (again from Greek), and so its name, “nephroid”. The cusps of a “nephroid” are spiky inward, hence a “nephroid” is an epicycloid. It doesn't seem to be a taboo to refer to a “nephroid” as a “2-cusped epicycloid”. An epicycloid with one cusp resembles the heart, and so a “cardioid”. The parametric equation seems to be easy to recognize due to the close relation to the polar equation for an ellipsis. It is $r = a(1 + \cos \theta)$ for a “cardioid” and $a = r(1 + \cos \theta)$ for an ellipsis (with eccentricity 1). We have failed in finding anatomic names for epicycloids with three cusps or more. However, we have found a stellar nomenclature for a 4-cusped object. An “astroid” is a hypocycloid (spiky outward) with four cusps. It is also called a tetracuspid, cubocycloid, or paracycle according to our reference CRC. Cubocycloid must have derived from cuboid which refers to a rectangular parallelepiped and also one of the tarsal bones.

In microlensing where the lenses are gravitationally bound multiple point masses, the metric is flat asymptotically, and the image of a source far away from the lens system is an unlensed image (or source itself) with $J = 1$. The critical curves are always closed curves, and the lens equation is always subcritical even when one includes dispersed medium of Galactic particle dark matter. One consequence is that the caustic curves are smooth closed curves punctuated by (spiky outward) cusps, or simply, hypocycloids. There is no 1-cusped hypocycloid or 2-cusped hypocycloid. After consulting the 2000-page reference, we may still feel in jeopardy how to extend the naming tradition of “-OID’s”. Or, is it “-ID’s”? The confusion between “-oid” and “-id” can arise from that a 3-cusped hypocycloid is called a “tricuspidOID”, and a 4-cusped hypocycloid is called a “tetracuspID” according to CRC. Considering that a 4-cusped hypocycloid is most commonly referred to as an “astroid” not “tetracuspID”, we conjecture that “tetracuspID” must have derived from dental terminology “cuspid” and lapsed attention to the particular characteristics of the points (cusps) in cycloids. A cuspid refers to a canine tooth which has a single projection point. A bicuspid refers to a premolar tooth which has two projection points. A tricuspid refers to a tooth that has three projection points and also a tricuspid valve. The suffix “-cuspid” seems to mean “pointed”. Cusps in cycloid are pointed in a particular manner where the tangent flips its sign. This explains why “tricuspid” is not a mathematical terminology for a 3-cusped hypocycloid. This explains why we referred to these objects as “some-form-of-oids” early on in this article. There doesn’t seem to be a usage of tetracuspID in dentistry or in anatomy.

Now, we discuss why chose the pattern of “number-oid” (“quadroid”) instead of “number-vertex-oid” (as in “tricuspid”) or “anatomy-oid” (as in “nephroid”). Following the “tradition” of borrowing anatomical names is excluded because of the likelihood of arbitrary number of cusps that may define caustic curves. So, the extension of imagination that may stem from “deltoid” and “cubocycloid” (and also “astroid” which is obviously an anatomy of a star if we look at a bright star in an HST frame) meets the dead end. We read that Euler studied a deltoid in 1745 in relation to an optical problem and also by Steiner in 1856, and a deltoid is referred to as Steiner’s hypocycloid in some literature. Despite all our intention to honor them, our option becomes limited to the pattern of “number-oid” or “number-vertex-oid”. Let’s consider epicycle and epicycloid to differentiate the two. An epicycle depicts a circular motion of an object around a center that moves along a larger circle, and the curve is smooth everywhere. An epicycloid depicts a circular motion of an object around the smaller circle that rolls on the larger circle, and the curve is smooth except at the cusps (the winding number is determined by the ratio of the two circles). It seems to be clear that “-oid” in “-oid” objects we discussed so far represents a particular resemblance to a cycle: curved and smooth like a circle but punctuated with points where the tangent flips its sign. In a binary lens, they are also simply closed curves. Thus, it is very clear that the number of cusps and the number of the smooth segments (or “sides”) of a caustic loop are the same, and it is sufficient to assign a number to specify the particular “-oid”. In a higher multiple point lens, a caustic loop can have winding number larger than 1. However, the winding number is a finite integer, and the number of vertices (or cusps) and the number of sides (or smooth segments) are the same. The caustic curve of a gravitational binary lens consists of one hexoid, two quadroids, or, one quadroid and two trioids.

- Preferences and Censors

We have reviewed that the intrinsic variables of the lens equation are angular variables. As in figure (2), we are free to choose a plane, set the distance scale, and mark all the variables, parameters, and objects we find fit from the lens equation. When $\ell \rightarrow 0$, the binary lens converges to a single lens, and the quadroid caustic around the center of mass contracts to a point caustic. So, “the single lens has a point caustic at the position of the lens.” We may draw the critical curve and caustic curve on the same (abstract) lens plane as we did in figure (2). The lens equation is a mapping from the chosen (abstract) lens plane to itself, or a mapping from the image plane to the source plane where the image plane refers to the (abstract) lens plane parameterized by the image position variable and the source plane refers to the (abstract) lens plane parameterized by the source position variable.

Now, is it confusing to call D_s -plane (the plane defined by the radial position of the radiation emission source) the source plane? Having understood that we only need to define one space (angular space) or a plane that corresponds to the angular space, it doesn’t seem to matter whatever the plane may be called. Once we know clearly what degrees of freedom we are manipulating through the lens equation, it doesn’t seem to be a confusing practice to let the beloved term “source plane” be used in both ways: based on the radial coordinate or based on the transverse coordinates. What is invariant seems to be that physics lies in relations not in nomenclatures. It is good to have distinguishable nomenclatures, but there is nothing holy (meaning leaving no room for scientific reasoning and fluidity) about the “source plane”. We find it sufficient to exercise a bit of contextual understanding to let the “source plane” enjoy the both definitions and let lensing colleagues keep their inertial tradition of describing lensing. Is it a law or a rule that the caustic curve lies on the source plane? We may choose to call the plane parameterized by the source position variable ω the ω -plane, perhaps from boredom or from a courtesy to leave the term “source plane” for the practitioners who are attached to the radial coordinate. Then, the caustic curve will lie on the ω -plane. So, it may constitute a proper question to ask if it is a law or a rule, and no paper should be shredded over this unsubstantiated dogma on wording.

Are newcomers to the field confused by the confusing usage of terminologies as Gould claims? Perhaps, not. We find it an unsubstantiated claim. Considering the suggestion of superluminal caustics, we would think that the proper route to resolve any confusion is for students to take a bit more than a few hours to understand lensing from first principles. We have found that the origin of the controversies lies in conceptual misunderstandings.

• Rapidly Rotating Black Holes and Optical Paths

Maxwell’s equations tell us how electromagnetic fields and matter interact. Lienard-Wiechert potentials satisfy the Maxwell’s equations with a moving charge. Special relativity is a property of the space time and so is the spin of a particle. When the velocity of the system becomes comparable to the speed of light, the space-like components of the fields become comparable to the time-like components. And, a novelty one may witness (as a student) from playing with the Lienard-Wiechert potentials is to see the electromagnetic wave propagate and actually carry the energy out to infinity. If we only pick out the time-component A_0 , we will be at a loss with the discrepancies with the measurements of the electric and

magnetic fields. Einstein field equations tell us how gravitational fields (or metric) and matter interact. When the velocity of the system becomes comparable to the speed of light, we expect that one should examine not only the time-time component (g_{00}) but also the other five components of the metric. We find it a baffling practice for ZG1199 to declare without substantiation that “retarding the Newtonian potential” (with additional factor 2 mentioned above) results in the “gravitational analog” that governs the behavior of the null geodesics as seen by an observer. We have no doubt that it will take a bit more than a few hours to examine the metric even at the post Newtonian level. However, we find it a worthy exercise to be carried out.

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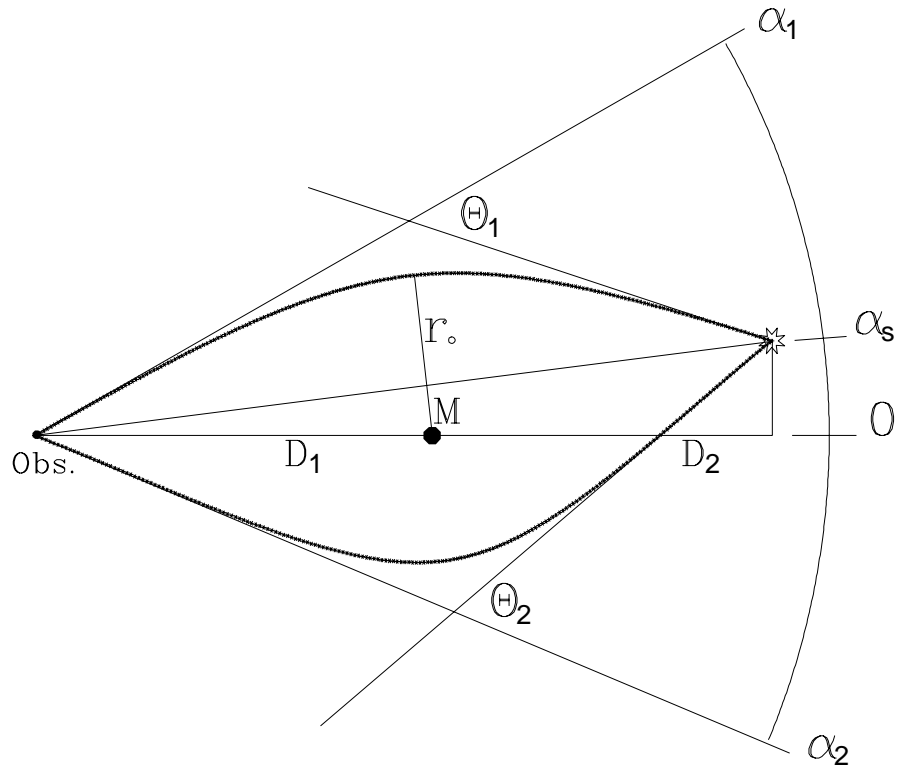


Fig. 1.—

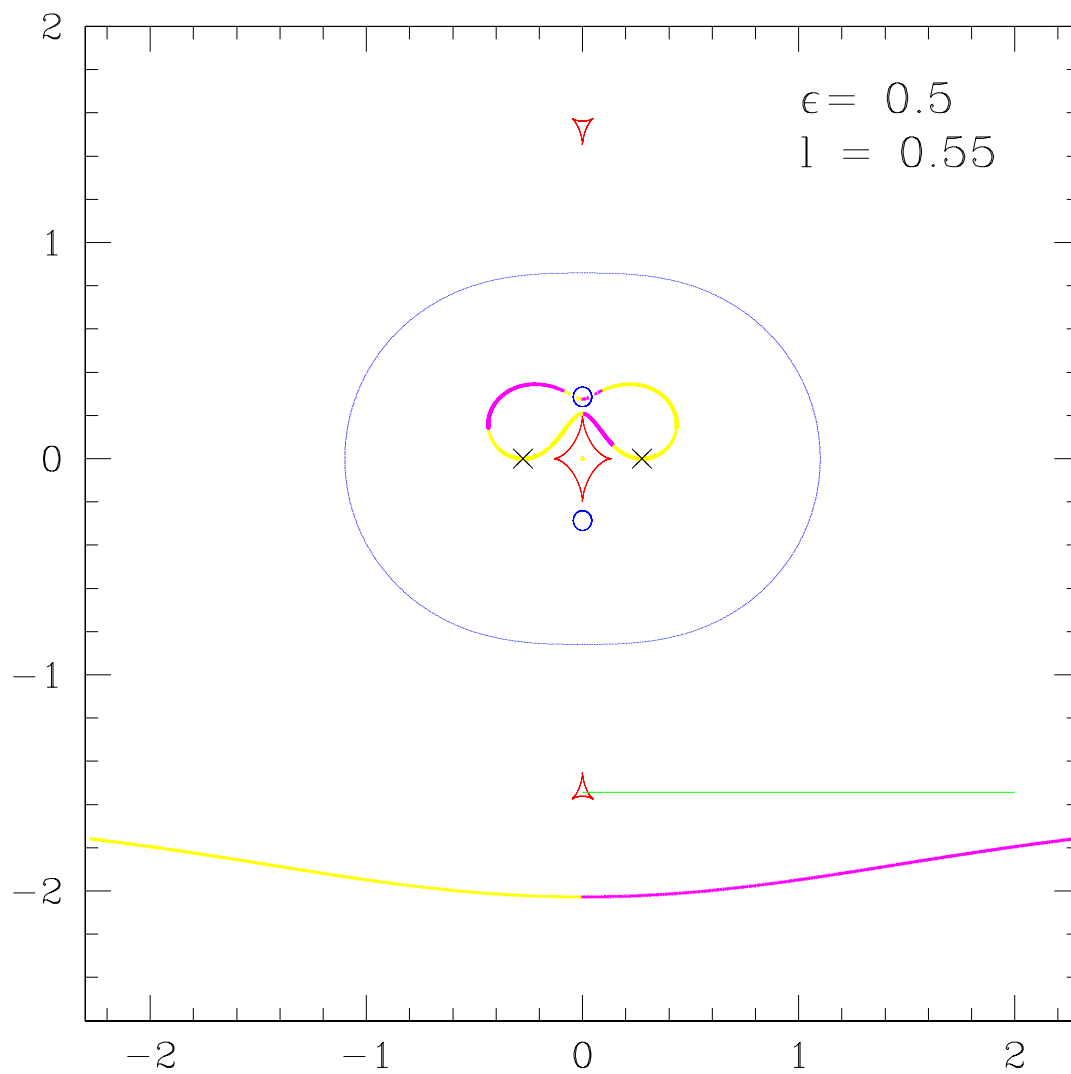


Fig. 2.—